

NO-A163 936

NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE  
OF COMPLEMENTARY SO. (U) GEORGIA INST OF TECH ATLANTA  
PRODUCTION AND DISTRIBUTION RESE. F A AL-KHAYVAL

1/1

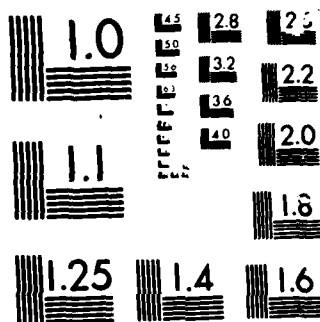
UNCLASSIFIED

NOV 85 PDRC-85-85 N00014-83-K-0147

F/G 12/1

NL

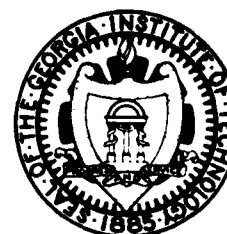




12

AD-A165 936

Georgia Institute  
of  
Technology



NECESSARY AND SUFFICIENT CONDITIONS FOR THE  
EXISTENCE OF COMPLEMENTARY SOLUTIONS AND  
CHARACTERIZATIONS OF THE MATRIX CLASSES  
Q AND  $Q_0$

by

Faiz A. Al-Khayyal

PDRC 85-05

DTIC  
SELECTED  
MAR 24 1986  
E

**PRODUCTION and  
DISTRIBUTION RESEARCH  
CENTER**

DTIC FILE COPY

SCHOOL OF INDUSTRIAL  
AND SYSTEMS  
ENGINEERING  
GEORGIA INSTITUTE OF TECHNOLOGY  
A UNIT OF THE UNIVERSITY SYSTEM  
OF GEORGIA  
ATLANTA, GEORGIA 30332

86 3 24 003

12

NECESSARY AND SUFFICIENT CONDITIONS FOR THE  
EXISTENCE OF COMPLEMENTARY SOLUTIONS AND  
CHARACTERIZATIONS OF THE MATRIX CLASSES  
 $Q$  AND  $Q_0$

by

Faiz A. Al-Khayyal

PDRC 85-05

School of Industrial and Systems Engineering  
Georgia Institute of Technology  
Atlanta, Georgia 30332

This work was supported by the Office of Naval Research under Contract No. N00014-83-K-0147 and by general research development funds provided by the Georgia Institute of Technology. Reproduction is permitted in whole or in part for any purpose of the U. S. Government.



Necessary and Sufficient Conditions for the Existence of Complementary  
Solutions and Characterizations of the Matrix Classes  $Q$  and  $Q_0$

Faiz A. Al-Khayyal  
School of Industrial and Systems Engineering  
Georgia Institute of Technology  
Atlanta, Georgia 30332-0205

Accession For	
NTIS GPA&I	<input checked="checked" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By _____	
Distribution/	
Availability Codes	
Dist	Special
A-1	



This work was supported in part by the Office of Naval Research under Contract No. N00014-83-K-0147 and by general research development funds provided by the Georgia Institute of Technology. Reproduction is permitted in whole or in part for any purpose of the U.S. Government.

## Abstract

### Necessary and Sufficient Conditions for the Existence of Complementary Solutions and Characterizations of the Matrix Classes $Q$ and $Q_0$

Faiz A. Al-Khayyal  
Georgia Institute of Technology

The linear complementarity problem has a solution if and only if there exists a right-hand-side vector of an associated dual linear program such that at least one LP solution falls inside a specific open set. This condition is used to give different characterizations of the matrix classes  $Q$  and  $Q_0$ . Moreover, they are used to derive verifiable sufficient conditions for a subclass of  $Q$ .

Key Words: Linear Complementarity Problem,  $Q$ -Matrices,  $Q_0$ -Matrices

Abbreviated Title: Conditions for LCP Solutions

This work was supported in part by the Office of Naval Research under Contract No. N00014-83-K-0147 and by general research development funds provided by the Georgia Institute of Technology. Reproduction is permitted in whole or in part for any purpose of the U.S. Government.

# Necessary and Sufficient Conditions for the Existence of Complementary Solutions and Characterizations of the Matrix Classes $Q$ and $Q_0$

Faiz A. Al-Khayyal  
Georgia Institute of Technology

## 1. Introduction

The linear complementarity problem is to find a real  $n$ -vector  $x$  such that

$$Mx+q \geq 0, x \geq 0, x^T(Mx+q) = 0, \quad (1)$$

where  $q$  is a known real  $n$ -vector and  $M$  is a known real  $n \times n$  matrix. The problem is generally denoted by the pair  $(q,M)$ . For given  $M$ , the feasible region of  $(q,M)$  is given by the convex polyhedron

$$S(q,M) = \{x \in \mathbb{R}^n : Mx+q \geq 0, x \geq 0\}. \quad (2)$$

When  $S(q,M) \neq \emptyset$ , the problem  $(q,M)$  and the vector  $q$  are said to be feasible. A feasible problem  $(q,M)$  does not necessarily have a solution. The main theorem of this paper provides a necessary and sufficient condition for problem  $(q,M)$  to have a solution. This result is then used to characterize two important matrix classes.

Numerous procedures have been proposed for solving problem  $(q,M)$ , including Lemke [11], Cottle and Dantzig [3], Murty [14], Eaves [7], Saigal [18], Garcia [8], and Karamardian [9]. More recently, Mangasarian [12,13], Cottle and Pang [4], Shiau [19], Pardalos and Rosen [16], Ramarao and Shetty [17], and Al-Khayyal [1] have developed different approaches for solving either (1) or an equivalent restatement of the

problem. While the preceding list is not exhaustive, with the exception of [1,16,17], all the methods assume structure in the matrix  $M$ . This note addresses two such classes of structured matrices.

The classes of matrices considered are

$$Q = \{M \in R^{n \times n} : q \in R^n \Rightarrow (q, M) \text{ has a solution}\}$$

and

$$Q_0 = \{M \in R^{n \times n} : q \in R^n \text{ and } S(q, M) \neq \emptyset \Rightarrow (q, M) \text{ has a solution}\}.$$

The class  $Q_0$  has also been denoted  $K$  in the literature (e.g., Pang [15]). Several authors have investigated these classes and have obtained full or partial characterizations of such matrices (see, e.g., Mangasarian [12], Cottle and Pang [4], Pang [15], Stone [20], Shiau [19], Doverspike and Lemke [6], Kelly and Watson [10], and Cottle, Randow and Stone [5]).

A geometric characterization of the members of  $Q_0$  is that the union of complementary cones of any matrix in  $Q_0$  is a convex set. The class  $Q$  is a subset of  $Q_0$  for which the union of complementary cones of any member in  $Q$  is  $R^n$ . In this note, we obtain new necessary and sufficient conditions for the classes  $Q$  and  $Q_0$  which have a different geometric interpretation. Verifiable conditions based on linear programming are provided which are sufficient for membership in  $Q$ .

## 2. The Conditions

Let  $Y$  denote the unit hypercube

$$Y = \{y \in R^n : 0 \leq y \leq e\},$$



where  $e = (1, 1, \dots, 1)^T$  is the summation vector. For fixed  $y \in Y$ , define the polyhedral set

$$V(y, M) = \{v \in R^n : M^T v < e - y, v > -y\},$$

and consider the linear program

$$\min\{q^T v : v \in V(y, M)\}. \quad (3)$$

Denote the solution set of this problem by

$$\operatorname{argmin}\{q^T v : v \in V(y, M)\},$$

and consider the following open polyhedral set

$$\hat{V} = \{v \in R^n : (M - I)^T v < e\}. \quad (4)$$

Theorem 1. Let  $q \in R^n$  and  $M \in R^{n \times n}$ . The problem  $(q, M)$  has a solution if and only if there exists a vector  $\bar{y} \in Y$  such that

$$\operatorname{argmin}\{q^T v : v \in V(\bar{y}, M)\} \cap \hat{V} \neq \emptyset. \quad (5)$$

Proof: (Necessity). Let  $\bar{x}$  solve the problem  $(q, M)$ . Let  $\bar{y} \in Y$  be defined as follows

$$\bar{y}_i = \begin{cases} 1 & \text{if } \bar{x}_i > 0 \\ 0 & \text{if } \bar{x}_i = 0. \end{cases}$$

It follows from linear programming duality that

$$0 \in \operatorname{argmin}\{q^T v : v \in V(\bar{y}, M)\} \cap \hat{V}$$

so that (5) is satisfied.

(Sufficiency). Let  $\bar{y} \in Y$  be such that (5) holds and let  $\bar{v} \in \operatorname{argmin}\{q^T v : v \in V(\bar{y}, M)\} \cap \hat{V}$ . Then by the Karush-Kuhn-Tucker Conditions there exist multipliers  $(\lambda, \xi)$  with  $\lambda \in S(q, M)$  and  $\xi = M\lambda + q$  such that

$$\begin{aligned} \lambda^T [e - \bar{y} - M^T \bar{v}] &= 0 \\ \xi^T [\bar{y} + \bar{v}] &= 0. \end{aligned} \tag{6}$$

It follows that  $\lambda^T (M\lambda + q) = \lambda^T \xi = 0$  so that  $\lambda$  solves  $(q, M)$ . For if this were not true, then there would exist an index  $i$  such that  $\lambda_i \xi_i = \lambda_i (M\lambda + q)_i > 0$ . By the complementary slackness condition (6), we must have

$$[e - \bar{y} - M^T \bar{v}]_i = 0 \quad \text{and} \quad [\bar{y} + \bar{v}]_i = 0.$$

Adding gives  $[e + (I - M)^T \bar{v}]_i = 0$  which contradicts the assump-

tion that  $\bar{v} \in \hat{V}$ .

Note that, when (5) holds for some  $\bar{y} \in Y$ , problem  $(q, M)$  has a solution and  $q$  is feasible. Thus, any infeasible  $q$  will violate (5) for all  $\bar{y} \in Y$ . The theorem can be restated to characterize the classes  $Q$  and  $Q_0$ .

Corollary 1. Let  $q \in R^n$  and  $M \in R^{n \times n}$ . The matrix  $M \in Q$  if and only if for each  $q$  there exists a vector  $\bar{y} \in Y$  such that (5) holds.

Corollary 2. Let  $q \in R^n$  and  $M \in R^{n \times n}$ . The matrix  $M \in Q_0$  if and only if for each feasible  $q$  there exists a vector  $\bar{y} \in Y$  such that (5) holds.

From the proof of the theorem, we see that when a complementary solution exists, there exists a  $y \in Y$  such that the origin must necessarily solve the linear program (3). The following example illustrates that this need not be the case when  $q$  is feasible and no complementary solution exists.

Example 1.  $M = \begin{bmatrix} 1/2 & 1 \\ 1 & -1/2 \end{bmatrix}, \quad q = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$

Here, we have  $S(q, M) = \{x: 1/2x_1 + x_2 + 1 > 0, x_1 - 1/2x_2 - 1 > 0, x_1 > 0, x_2 > 0\} \neq \emptyset$ , but there is no  $y \in Y$  for which the origin solves problem (3). In fact, the optimal solution to the linear program (3) is not in  $\hat{V}$  for any  $y \in Y$ .

Formalizing these observations, let  $cl \hat{V}$  denote the closure of the set  $\hat{V}$  and let  $\partial(cl \hat{V})$  denote the boundary of the set  $cl \hat{V}$ . We have the

following consequence of the theorem.

Corollary 3. Let  $q \in \mathbb{R}^n$  and  $M \in \mathbb{R}^{n \times n}$ . The problem  $(q, M)$  does not have a solution if and only if for every  $\bar{y} \in Y$ ,

$$\operatorname{argmin}\{q^T v : v \in V(\bar{y}, M)\} \subseteq \partial(\operatorname{cl} \hat{V}). \quad (7)$$

Proof: From Theorem 1, it follows that problem  $(q, M)$  does not have a solution if and only if for every  $\bar{y} \in Y$

$$\operatorname{argmin}\{q^T v : v \in V(\bar{y}, M)\} \cap \hat{V} = \emptyset. \quad (8)$$

However, it is clear that

$$\operatorname{argmin}\{q^T v : v \in V(\bar{y}, M)\} \subseteq \operatorname{cl} \hat{V}. \quad (9)$$

The result follows since (8) and (9) are true if and only if (7) holds.

For completeness, this will be stated in terms of the classes  $Q$  and  $Q_0$ .

Corollary 4. Let  $q \in \mathbb{R}^n$  and  $M \in \mathbb{R}^{n \times n}$ . The matrix  $M \notin Q$  if and only if there exists a  $q$  such that (7) holds for every  $\bar{y} \in Y$ .

Corollary 5. Let  $q \in \mathbb{R}^n$  and  $M \in \mathbb{R}^{n \times n}$ . The matrix  $M \notin Q_0$  if and only if there

exists a feasible  $q$  such that (7) holds for every  $\bar{y} \in Y$ .

New sufficient conditions for membership in  $Q$  can also be derived from Theorem 1. Let

$$V' = \{v \in R^n: M^T v \leq e, v \geq -e\}. \quad (10)$$

Corollary 6. If  $V' \subset \hat{V}$  then  $M \in Q$ .

Proof: For every  $y \in Y$ , we have  $V(y, M) \subseteq V'$ . Hence, for any  $q \in R^n$ , if  $V' \subset \hat{V}$  then  $\operatorname{argmin}\{q^T v: v \in V(y, M)\} \subset \hat{V}$  so that (5) is satisfied. Thus every  $q \in R^n$  yields a solution to problem  $(q, M)$  so that  $M \in Q$ .

The condition in Corollary 6 is rather strong since it implies that (5) is satisfied for all  $y \in Y$ . Nevertheless, it includes some  $P$ -matrices as illustrated by the following example.

Example 2.  $M = \begin{bmatrix} 1 & 1/2 \\ -1/2 & 1 \end{bmatrix},$

$$V' = \{(v_1, v_2): 2v_1 - v_2 \leq 2, v_1 + 2v_2 \leq 2, v_1 \geq -1, v_2 \geq -1\}$$

$$\subset \{(v_1, v_2): v_1 \leq 2, v_2 \geq -2\} = \hat{V}.$$

On the other hand, the next example shows that there are  $P$ -matrices that violate the condition  $V' \subset \hat{V}$ .

Example 3.  $M = \begin{bmatrix} 1/2 & 1/2 \\ -2 & 1 \end{bmatrix},$

$$V' = \{(v_1, v_2): v_1 - 4v_2 \leq 2, v_1 + 2v_2 \leq 2, v_1 \geq -1, v_2 \geq -1\},$$

$$\hat{V} = \{(v_1, v_2): v_1 + 4v_2 \geq -2, v_1 \leq 2\}.$$

Here,  $V' \not\subset \hat{V}$  since  $(-1, -3/4)$  is in  $V'$  but not in  $\hat{V}$ .

The preceding corollary suggests checkable conditions to test whether or not  $V' \subset \hat{V}$ . First, recall that a nonzero vector  $d$  is called a direction of a set if the ray emanating from every point in the set in the direction  $d$  is entirely contained in the set (see, e.g., Bazaraa and Jarvis [2], Chapter 2). Let  $D(V')$  and  $D(\hat{V})$  denote the set of directions of  $V'$  and  $\hat{V}$ , respectively. These sets define the polyhedral cones

$$D(V') = \{d \neq 0: M^T d \leq 0, d \geq 0\}$$

and

$$D(\hat{V}) = \{d \neq 0: (M-I)^T d \leq 0\}.$$

Note that

$$D(V') \subseteq D(\hat{V}). \quad (11)$$

Theorem 2. The set  $V'$  is a subset of  $\hat{V}$  if and only if for every extreme point  $\bar{v}$  of  $V'$ ,

$$\text{either } (M^T \bar{v})_i = 1 \quad \text{and } \bar{v}_i > 0 \quad (12)$$

$$\text{or } (M^T \bar{v})_i < 0 \quad \text{and } \bar{v}_i = -1$$

for all  $i = 1, 2, \dots, n$ .

Proof: (Necessity). Suppose  $V' \subset \hat{V}$  and let  $\bar{v}$  denote an arbitrary extreme point of  $V'$ . Since  $\bar{v}$  is an extreme point, then at least  $n$  linearly independent constraints of the  $2n$  constraints of  $V'$  must be binding at  $\bar{v}$ ; thus,

$$\text{either } (M^T \bar{v})_i = 1 \quad \text{or } \bar{v}_i = -1 \quad (13)$$

for all  $i = 1, 2, \dots, n$ . But  $V' \subset \hat{V}$  so that

$$(M^T \bar{v})_i - \bar{v}_i < 1 \quad (14)$$

for all  $i = 1, 2, \dots, n$ . Clearly, (13) and (14) imply (12).

(Sufficiency). Assume that the set  $V'$  has  $k$  extreme points  $\{v^1, v^2, \dots, v^k\}$  and  $\ell$  extreme directions  $\{d^1, d^2, \dots, d^\ell\}$ . By (12), we have

$$(M-I)^T v^j < e \quad (15)$$

for all  $j = 1, 2, \dots, k$ . Since  $V'$  is a convex polyhedral set, then  $v \in V'$  if and only if there exist nonnegative vectors  $(\lambda, \mu)$  with  $e^T \lambda = 1$  such that

$$v = \sum_{j=1}^k \lambda_j v^j + \sum_{j=1}^l \mu_j d^j.$$

Hence,

$$(M-I)^T v = \sum_{j=1}^k \lambda_j (M-I)^T v^j + \sum_{j=1}^l \mu_j (M-I)^T d^j < e.$$

The last inequality follows from (15) and (11) and the conditions  $e^T \lambda = 1$  and  $\mu > 0$ . Therefore  $v \in \hat{V}$ , which completes the proof.

### 3. Summary and Conclusions

This note presents new necessary and sufficient conditions for the existence of a solution to the linear complementarity problem. These translate to similar conditions for the classes of matrices  $Q$  and  $Q_0$  and for their complements. In particular, Theorem 1 (Corollary 2) states that existence of a solution to the linear complementarity problem (membership in  $Q_0$ ) is equivalent to the existence of a solution to the linear program (3), for some fixed  $y \in Y$ , which is also in the open set  $\hat{V}$  given by (4). From this follows the interesting condition on membership in the complement of  $Q_0$ , which is equivalent to the existence of a feasible  $q$  such that every solution of the linear program (3), for every fixed  $y \in Y$ , is on the boundary of the closure of  $\hat{V}$ .

While these conditions are not easy to verify, they do provide a new framework from which new results can be derived. For example, Corollary 6 provides a sufficient condition for membership in  $Q$ . This condition can be verified by using the test suggested by Theorem 2. This test involves generating all the extreme points of  $V'$  and checking each



against (12). Note that to establish that a matrix is not in  $Q$  would require generating all the extreme points only in the worst case. However, to confirm that a matrix is in  $Q$ , all extreme points must be generated.

Acknowledgements. The author wishes to thank Richard E. Stone, Donna Crystal, and John Vande Vate for their valuable comments on an earlier draft of this paper.

## REFERENCES

- [1] Al-Khayyal, F. A., "An implicit enumeration procedure for the general linear complementarity problem," Industrial and Systems Engineering Report Series No. J-83-7, Georgia Institute of Technology, Atlanta, Georgia 1983.
- [2] Bazaraa, M. S., and Jarvis, J. J., Linear Programming and Network Flows, John Wiley & Sons, New York, 1977.
- [3] Cottle, R. W., and Dantzig, G. B., "Complementary pivot theory of mathematical programming," Linear Algebra and Its Applications 1 (1968) 103-125.
- [4] Cottle, R. W., and Pang, J. S., "On solving linear complementarity problems as linear programs," Mathematical Programming Study 7 (1978) 88-107.
- [5] Cottle, R. W., Rando, R. von, and Stone, R. E., "On spherically convex sets and Q-matrices," Linear Algebra and Its Applications 41 (1981) 73-80.
- [6] Doverspike, R. D., and Lemke, C. E., "A partial characterization of a class of matrices defined by solutions to the linear complementarity problem," Mathematics of Operations Research 7 (1982) 272-294.
- [7] Eaves, B. C., "The linear complementarity problem," Management Science 17 (1971) 612-634.
- [8] Garcia, C. B., "Some classes of matrices in linear complementarity theory," Mathematical Programming 5 (1973) 299-310.
- [9] Karamardian, S., "The complementarity problem," Mathematical Programming 2 (1972) 107-129.
- [10] Kelly, L. M. and Watson, L. T., "Q-matrices and spherical geometry," Linear Algebra and Its Applications 25 (1979) 175-189.
- [11] Lemke, C. E., "Bimatrix equilibrium points and mathematical programming," Management Science 4 (1965) 681-689.
- [12] Mangasarian, O. L., "Characterization of linear complementarity problems as linear programs," Mathematical Programming Study 7 (1978) 74-87.
- [13] Mangasarian, O. L., "Simplified characterizations of linear complementarity problems solvable as linear programs," Mathematics of Operations Research 4 (1979) 268-273.
- [14] Murty, K. G., "Note on a Bard-type scheme for solving the linear complementarity problem," Opsearch 11 (1974) 123-130.

- [15] Pang, J. S., "A note on an open problem in linear complementarity," Mathematical Programming 13 (1977) 360-363.
- [16] Pardalos, P. M., and Rosen, J. B., "Global optimization approach to the linear complementarity problem," Technical Report 84-37, Computer Sciences Department, University of Minnesota, Minneapolis, Minnesota, 1984.
- [17] Ramarao, B., and Shetty, C. M., "Application of disjunctive programming to the linear complementarity problem," Naval Research Logistics Quarterly 31 (1984) 589-600.
- [18] Saigal, R., "On the class of complementary cones and Lemke's algorithm," SIAM J. Appl. Math. 23 (1972) 46-50.
- [19] Shiau, T. H., "Iterative linear programming for linear complementarity and related problems," Computer Sciences Technical Report No. 507, University of Wisconsin, Madison, Wisconsin, 1983.
- [20] Stone, R. E., "Geometric aspects of the linear complementarity problem," Ph.D. dissertation, Technical Report SOL 81-6, Systems Optimization Laboratory, Stanford University, Stanford, California, 1981.

**END**

**FILMED**

4-86

**DTIC**